INFINITESIMALS ARE TOO SMALL FOR COUNTABLY INFINITE FAIR LOTTERIES

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Abstract. We show that infinitesimal probabilities are much too small for modeling the individual outcome of a countably infinite fair lottery.

1. Introduction

Consider a fair lottery with an infinite number of tickets, numbered by the set $\mathbb{N} = \{1, 2, ...\}$ of natural numbers. What is the probability of getting any particular result, say 1 or 94880? It can't be any real number $r > 0$ on pain of violating the total probability and finite additivity (there is no hope of countable additivity here) postulates, since all outcomes must have the same probability so that $P(\{1, ..., k\}) = P(\{1\}) + \cdots + P(\{k\}) = kP(\{1\}) = kr$ and if k is sufficiently large, then kr will exceed 1. On the other hand, setting the probability of particular outcomes to zero, violates Bayesian regularity—the thesis that contingent propositions should have probabilities strictly between 0 and 1—and is counterintuitive because then the probability of getting either 1 or 2, which will also have to be zero by finite additivity, is no greater than the probability of getting 1. Furthermore, assigning zero to the probability of each particular outcome yields a merely finitely additive probability measure, which results in paradoxes such as reasoning to a foregone conclusion or paying not to receive information (Kadane, Schervish and Seidenfeld, 1996).

This has led a number of people (for instance, Wenmackers and Horsten, forthcoming) to the idea that we should let the probability of a particular outcome of the lottery be an infinitesimal number δ in some field like the hyperreals that extends the real numbers. Unfortunately, this still is subject to paradox. Because the standard part[1] of the measure is still merely finitely additive, we still have the paradoxes of reasoning to a foregone conclusion and paying not to receive information, We still have a non-conglomerable probability measure (this follows by applying the theorems of Schervish, Seidenfeld and Kadane [1984] to the standard part of the measure); the non-conglomerability can spill over into paradoxical results about matters

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[1]For any finite number x in an ordered field F extending the reals, there is a standard part $\mathrm{St}\, x$, which is the unique real number such that $x - \mathrm{St}\, x$ is infinitesimal (we can let $\mathrm{St}\, x = \sup\{y \in \mathbb{R} : y < x\}$).

other than infinite lotteries (Pruss, 2012); and no reasonable expected value function here satisfies domination (Pedersen, manuscript).

But while those arguments show *that* the use of hyperreal infinitesimals does not solve the problem, they raise an explanatory question: Why doesn't it? After all, intuitively hyperreal infinitesimals seem like just the right tool. I shall offer two arguments in favor of a diagnosis of a *fundamental* problem with assigning hyperreal infinitesimal probability here: such infinitesimals are infinitely "too small" to give plausible probabilities of individual outcomes in a countably infinite lottery.

There certainly are finitely additive measures that assign the same infinitesimal to every outcome of a lottery (e.g., the method of Bernstein and Wattenberg, 1969, already readily adapts to yield this: see Wenmackers and Horsten, forthcoming). But the problem is that of the many such measures, none can assign a *correct* infinitesimal. For any finite positive γ, and any finitely additive probability measure P that assigns probability δ to every outcome in a sample space Ω, there is a finitely additive probability measure P^γ that assigns probability $\gamma\delta$ to every outcome.[2] So if we are to assign an infinitesimal probability to the outcomes of an infinite fair lottery, we need to choose a *particular* infinitesimal probability to assign. But which one (this worry was already raised by Elga, 2004)? Alas, as we shall see, no hyperreal infinitesimal probability is sufficiently large for a lottery on $\mathbb{N}$.

Alan Hájek (2011) has argued that if we assign infinitesimal probabilities to contingent probabilities to save Bayesian regularity, then an arms race results, because we can then ask for probabilities on large hyperreal domains, which will require higher-order infinitesimals, and so on. Our arguments show that there is no arms race: the Bayesian simply loses in the case of a countably infinite fair lottery, as infinitesimal probabilities are simply not suited to restoring regularity.

Both of our arguments will use the notion of a positive hyperreal a being infinitely larger than another positive hyperreal b. This will be written "$a >> b$" and is defined to hold whenever a/b is infinite, or, equivalently, b/a is infinitesimal. The first argument is perhaps more of an intuition pump, while the second gives a more precise diagnosis of what has gone wrong.

2. Argument one

Suppose Jim denies finite additivity of probabilities and assigns the probabilities 1/10, 1/100 and 1/1000, respectively, to the outcomes of a lottery that has three tickets and a guarantee that exactly one wins. It would beg the question against Jim to say that his probabilities don't add up to 1.

[2]Let $P^\gamma(A) = \operatorname{St} P(A) + \gamma(P(A) - \operatorname{St} P(A))$. Then it is easy to see that P^γ is a finitely additive probability measure if P is, and $P^\gamma(\{n\}) = \gamma P(\{n\}) = \gamma\delta$, since $\operatorname{St} P(\{n\}) = 0$ as $P(\{n\})$ is infinitesimal.

However, if we could get Jim to accept that there can be a perfectly uncontroversial lottery with three tickets that assigns the probabilities 1/2, 1/4 and 1/4, respectively, to the outcomes, we could then observe that the probabilities that Jim assigned are each much smaller than the probabilities assigned by the uncontroversial lottery. But surely there could not be a lottery with the same tickets as Jim's alleged lottery, and yet still with every ticket being much more likely to win than in Jim's lottery. That every ticket in Jim's lottery is much less probable than in the uncontroversial lottery is a sign that Jim has assigned much too small probabilities to the outcomes.

Now, suppose that Jim has assigned infinitesimal probability $\delta_n > 0$ to the probability $P_1(\{n\})$ of each outcome of an infinite fair lottery (this argument does not need the infinitesimal probabilities to be all the same). We could try to fault Jim's assignment for violating countable additivity, because the sum of countably many infinitesimals is undefined, but that would likely beg the question.

Nonetheless, we can extend the argument in the three ticket case. Consider the uncontroversial classical infinite lottery with individual outcome probabilities $P_2(\{n\}) = 2^{-n}$. This probability assignment extends to an entirely unparadoxical countably additive probability on $\mathbb{N}$. Observe now that for every individual outcome $\{n\}$, we have $P_1(\{n\}) = \delta_n << 2^{-n} = P_2(\{n\})$. In other words, our controversial lottery assignment assigns an infinitely smaller probability to *each* outcome than the unproblematic classical lottery. And just as in the case of the problematic three-outcome lottery, this gives us good reason to say that every ticket in the lottery with infinitesimal outcome probabilities has been assigned a probability that is much too small, indeed infinitely too small.

A variant formulation is to note that assigning probability 10^{-n} to the nth ticket, for every n, will assign too small probabilities to the tickets ($1/10 + 1/100 + 1/1000 + \cdots$ is significantly smaller than 1). But then *a fortiori* assigning an infinitesimal to every ticket will assign too small probabilities to the tickets, since an infinitesimal is smaller than 10^{-n} no matter what natural number n we choose.

This argument works without change for infinitesimals in any non-Archimedean field extending the real numbers.

3. Argument two

3.1. **The basic argument.** Suppose Jim assigns probability 1/100 to each outcome of a fair lottery where it is guaranteed that exactly one of three tickets wins, but Jim denies finite additivity. If we could get Jim to admit that the probability of each outcome in a fair lottery with four tickets is 1/4, then we should be able to convince him that the probabilities he assigned in the three ticket case are much too small, since the probability of winning a fair lottery does not go up as we increase the number of tickets. We shall now run a version of this argument in the infinitesimal case.

Let ${}^*\mathbb{N}$ be the hypernaturals, i.e., the ordinary natural numbers plus the infinite positive hyperintegers. For any $n \in {}^*\mathbb{N}$, let $[n] = \{k \in {}^*\mathbb{N} : 1 \leq k \leq n\}$ be the set of the first n hypernaturals.

The structure of the basic argument will be this. I will show that for any infinitesimal $\delta > 0$, there is an infinitesimal β which is infinitely larger than δ and an infinite set J of which $\mathbb{N}$ is a subset such that when we consider an infinite fair lottery on J, then β is *clearly* the right infinitesimal probability to assign to each particular outcome, assuming we are going to assign infinitesimal probabilities at all. But the probability of winning a fair lottery does not go up as we increase the set of possible winners, so if δ is infinitely smaller than the probability of winning the larger lottery, then δ is infinitely too small to give a plausible representation of the probability of winning the smaller lottery. Thus every infinitesimal is infinitely too small to model a countably infinite lottery.

So suppose that P_1 is a finitely additive probability on $\mathbb{N}$ that assigns an infinitesimal $\delta > 0$ to every singleton and represents our countably infinite fair lottery. Choose a hypernatural infinite number M small enough that $\delta << 1/M$.[3]

Now consider an infinite fair lottery on the set $[M]$ of all hypernaturals less than or equal to M. Since M is infinite, this set is infinite. Once we have accepted the idea that infinite fair lotteries should have infinitesimal outcome probabilities, we will want to assign an infinitesimal probability for each of the outcomes in the set $[M]$. And while in the case of an infinite fair lottery on $\mathbb{N}$ it is not clear *which* infinitesimal should represent the probability of a particular outcome, there is an obvious answer as to what that infinitesimal probability here should be: $P_2(\{n\}) = 1/M$ for all $n \in [M]$.

But now observe that $\mathbb{N}$ is a proper subset $[M]$. For any $n \in \mathbb{N}$ is finite, and hence less than the infinite number M, while the infinite hypernatural M is in $[M]$ but not in $\mathbb{N}$. Thus we have the following situation: P_1 which allegedly models a fair lottery on the smaller set $\mathbb{N}$ assigns an infinitesimal δ to every member of $\mathbb{N}$, while P_2 which with much greater plausibility models a fair lottery on the larger set $[M]$ assigns an infinitely bigger infinitesimal $\beta = 1/M$ to every member of $[M]$. Moreover, not only is the set $[M]$ larger than $\mathbb{N}$, but it is *much* larger, because $\mathbb{N}$ is countably infinite, while $[M]$ is uncountably infinite (this is a well-known fact in the special case of the hyperreals, but an elementary proof in a more general setting is given in the Appendix).

If this pair of probability assignments were correct, then we could imagine you bought a ticket in the fair lottery with countably many outcomes, and now you are offered to switch to a new fair lottery which adds uncountably more tickets but still has one and the same prize. The switch would obviously

[3]For instance, we might let M be the unique hypernatural number such that $M \leq 1/\sqrt{\delta} < M+1$. Then $\delta/(1/M) = M\delta \leq \sqrt{\delta}$. But $\sqrt{\delta}$ is infinitesimal if δ is, so $\delta << 1/M$.

be irrational, but according to the above assignments, you infinitely increase your probability of getting the prize by switching.

Given the correctness of the assignment of probability $1/M$ in the second lottery, the lottery with fewer tickets should assign at least $1/M$. But in fact it assigns δ, which is infinitely less than $1/M$. Thus, the lottery on the naturals has been assigned infinitely too small a value for each possible outcome when it was assigned an infinitesimal in each case.

Here's perhaps a more intuitive way to see what has gone wrong. Suppose for simplicity that our infinitesimal δ is of the form $1/N$ for some hypernatural N. This might not be exactly true, but no matter what δ is, $1/\delta$ will be within distance 1 of *some* hypernatural, and then δ will be very close to the reciprocal of that hypernatural. Now, if $\delta = 1/N$, then δ correctly represents the individual outcome probabilities for a fair lottery on $[N]$. But $\mathbb{N} \subset [N]$ and in fact $[N]$ is a much larger set. We can see that it is much larger in two ways. First, there is a set $[M]$ that contains $\mathbb{N}$ such that $M << N$ (for instance, just let M be the result of rounding $\sqrt{N}$ to the nearest hypernatural). Second, $[N]$ is uncountable while $\mathbb{N}$ is countable. So if δ is the right value for an outcome of a fair lottery on $[N]$ (or, in the case where $1/\delta$ is not hypernatural, is very close to that value), δ is much too small for an outcome of a fair lottery on $\mathbb{N}$.

3.2. **Other extensions of the reals.** For simplicity, our argument was given in terms of the hyperreals, but the main argument extends directly to the surreals and to any other totally ordered field extensions of the reals which contain a plausible analogue to the hypernaturals (in that direction, see Mourges and Ressayre, 1993, and the Appendix). But it is only where there is an analogue of the notion of an integral number (which then yields the hypernaturals as the positive integral numbers) that infinitesimal probabilities in a fair lottery make plausible sense, since we would expect the probability of an outcome of a fair lottery on a set Ω to be the reciprocal of some sort of a *count* of the number of elements of Ω, and a count should be in an appropriate sense an integral number.

4. Final remarks

We have seen that infinitesimals in a totally ordered field extension of the reals are too small to be the probabilities of a particular ticket in a countably infinite fair lottery winning.

An anonymous reader has suggested that the arguments also show that *zero* is also too small to be the probability of a particular ticket winning. If so, then we now have an argument that *no probability* can be assigned to an individual outcome of a countably infinite lottery. For zero and an infinitesimal are too small, while every positive real is too big as we saw by finite additivity at the beginning of the paper. The reader is right in the case of our first argument: 0 is indeed infinitely smaller than every outcome probability of the uncontroversial lottery with $P(\{n\}) = 2^{-n}$, and so for the same reason that an infinitesimal is too small an outcome probability for the fair lottery, so is zero. But it is not clear that the second argument can be extended to show that zero is too small. For our second argument assumes that infinitesimals are the right probabilities to assign in infinite lotteries, and concludes *given this* that the lottery on $[M]$ should have individual outcome $1/M$. But someone holding out for a zero-probability answer might just say that the individual outcome probability for *any* infinite fair lottery, including one on $[M]$, should be zero.[4]

Appendix: The uncountability of $[M]$

One of our arguments used the fact that $[M]$, i.e., the set of hypernaturals less than or equal to M, is an uncountable set when M is infinite. This follows from standard facts about hyperreals (e.g., Lemma 11.2.2 of Corbae, et al., 2009), but it is worth giving a simple proof that works not just in the case of the hyperreals, but for any extension of the reals that has an analogue of the hyperintegers.

Let F be any totally ordered field containing the real numbers as an ordered subfield. Say that a subset Z of F is a set of *F-integers* provided that (a) $0 \in Z$ and (b) for every positive $x \in F$, there is a unique $n \in Z$ such that $n \leq x < n + 1$. This unique n is called *the integer part* of x (see Mourges and Ressayre, 1993), and we will write $n = \lfloor x \rfloor_Z$. Write $[M]_Z = \{n \in Z : 0 < n \leq M\}$. Say that $M \in F$ is infinite provided that $x < |M|$ for all real x (thus, M is infinite if and only if $1/M$ is infinitesimal).

Theorem. *If $M \in F$ is infinite and positive, then $[M]_Z$ has at least the cardinality of the continuum, and in particular is uncountable.*

Proof. Write $(0, 1)$ for the interval of all *real* numbers strictly between 0 and 1. Let $A = \{\lfloor xM \rfloor_Z : x \in (0, 1)\}$. Observe that $(0, 1)$ and A have the same cardinality. For let $f(x) = \lfloor xM \rfloor_Z$. This is a function from $(0, 1)$ onto all of A. To get that $(0, 1)$ and A have the same cardinality, we need only prove that f is one-to-one. To do this, suppose $f(x) = f(y)$ for x and y in $(0, 1)$. Then $\lfloor xM \rfloor_Z = \lfloor yM \rfloor_Z$ and there is an $n \in Z$ such that $n \leq xM < n + 1$ and $n \leq yM < n + 1$. It follows that $|xM - yM| < 1$. Thus, $|x - y|M < 1$ as M is positive, and so $|x - y| < 1/M$. But if M is infinite, then $1/M$ is infinitesimal, and the non-negative real number $|x - y|$ is less than that infinitesimal. But the only non-negative real number less than an infinitesimal is zero, so $|x - y| = 0$ and hence $x = y$. Thus, f is one-to-one and onto, and so A has the same cardinality as $(0, 1)$, i.e., the cardinality of the continuum.

But A is a subset of $[M]_Z$ as is easy to check (every member of A is a non-negative member of Z and less than M; also, $0 \notin A$ since M's being infinite guarantees that xM is infinite for all $x \in (0, 1)$, and hence that $\lfloor xM \rfloor_Z$ is

[4]The author is grateful to three anonymous readers for comments that have improved this paper.

also infinite for all $x \in (0,1)$). Hence, $[M]_Z$ has at least the cardinality of the continuum, and in particular is uncountable. $\square$

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